Sequential Constraint Estimation: Implementation Modifications

Allen R. Stubberud
University Of California Irvine
Department Of Electrical Engineering And Computer Science
Henry Samueli School Of Engineering
Irvine, CA 92697-2625
United States of America
arstubbe@uci.edu

Stephen C. Stubberud
The Boeing Company
3370 Miraloma Avenue
P.O. Box 3105
Anaheim, CA 92803-3105
United States of America
scstubberud@ieee.org

Abstract - *In recent works, the first author has developed a constraint estimation technique that can be applied to the kinematic tracking problem. This a posteriori technique maintains an unconstrained estimate based solely on the measurements while the constrained estimate is computed, if necessary, at each time step. This technique has been expanded to incorporate nonlinear systems and time-varying systems. In this paper, we modify the original approach to accommodate several implementation issues that have arisen in the discussions about this work. The first modification of this approach is to utilize a probability-based estimator, the extended Kalman filter, as the unconstrained estimator as opposed to the deterministic unconstrained estimator in this work. By showing a minimal variation in performance of the algorithm, the constrained estimator can be shown to be a simple addition to standard tracking estimation routines.*

Keywords: Constraint estimation, sequential estimator, extended Kalman filter, tracking, comparative implementation.

1 Introduction

One approach to enhancing the kinematic fusion solution in target tracking is to improve the motion model of the tracked target. Over the years, a number of techniques have been proposed to provide an improved model of target motion. Techniques have included circular motion models that vary the radial parameter, neural networks that adapt while tracking to model the maneuver [1,2], and the popular variable structure interacting multiple model (VS-IMM) technique [3] that uses a fixed number of defined motion models to create an interpolated solution to the motion model of the target.

Another approach to improving target tracking is to incorporate the constraints that exist in the target tracking model. The location of the motion of a target often is constrained physically. This can be the result of terrain, vehicle classification, and/or doctrine. In [4-6], a technique that incorporates constraints into the estimation process has been developed. A unique property of this technique is that the constraint is incorporated *a posteriori* to the incorporation of the measurement to the estimate. Another property that arose out of the mathematical development of this approach is that the constrained estimate at a given time is dependent only on the unconstrained estimate at the same time.

The implementation and testing of this technique has demonstrated its usefulness. However, the two aforementioned properties have given light to its potential as an acceptable technique that can be applied with other estimators. In this paper, we look at the implementation of the constrained estimator. We first examine the incorporation of the standard unconstrained estimator, the extended Kalman filter (EKF), into the process. Second, we address the issues associated with multiple constraints.

In Section 2, we summarize the constrained estimation technique as it has been derived. We follow this in Section 3 with the modification of the implementation by using the EKF as the unconstrained estimator. This is followed by a comparison between the two approaches.

2 Sequential State Estimator

In this section, a sequential estimation routine that computes an estimate of the kinematic state of a target is presented. This technique was first developed as a result of a discussion about the effects of constraint violations by a tracking system on the operators. While the measurements may drive the track to be placed in a nonfeasible solution, the operator will look at the underlying tracking system as erroneous overall. As seen in Figure 1, while the tracks clearly indicate that target's location and its associated error ellipse clearly are within reason, an operator would prefer to see the target continue on the path. Simply placing the target on the constraint without a mathematical foundation is ad hoc and could easily be challenged. Also, modification of the Kalman filter with constraint logic can remove the mathematical underpinnings and optimality associated with the technique. Thus, we began to look at addressing the problem with a new approach.

The state's dynamics are assumed to be modeled by a nonlinear time-variable discrete-time difference equation.

In addition, the target's state estimates are subject to an instantaneous constraint, that is, at each sample time the state estimate is constrained to lie in a given region of the state space.The nonlinear estimator is an extended version of a linear time-variable weighted least squares sequential estimator, that is, the nonlinear estimator is developed by using a linearization process in which the target's defining nonlinear functions are expanded in a Taylor series and the linear term of the Taylor series is retained after evaluating it at the last best available estimates of the target states and measurements. A detailed development of the basic constrained sequential linear and nonlinear estimators are available in references [4-6]. A brief outline of that development is given below.

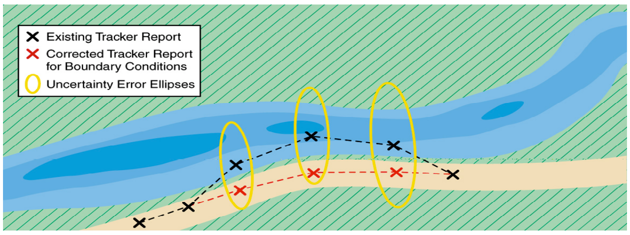

Figure 1: Constraints may be violated by the estimator which can cause operators to question all solutions of tracking system.

In the following abbreviated development of the constrained sequential state estimator of a nonlinear system, the first step is the development of an unconstrained weighted least squares sequential state estimator for a time-variable linear system. This is followed by the development of constrained weighted least squares sequential state estimator for the same system using the results from the unconstrained estimator. Finally, extended versions of the unconstrained and constrained estimators are developed for a nonlinear system.

Assume that a dynamic system is described by a linear, time-variable difference equation

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{w}(k), \quad k \quad 0 \tag{1}$$

where $\mathbf{x}$ is the n-dimensional state vector, $\mathbf{w}$ is an n-dimensional vector of unknown noises, $\mathbf{A}(k)$ is the time-variable system matrix of dimension $n \quad n$ and k is a nonnegative integer representing discrete time. Note that since the mathematical development that follows is deterministic, a probabilistic description of the noise $\mathbf{w}$ is not necessary. Assume the state of the system is measured through a noisy, linear, time-variable transformation

$$\mathbf{z}(k) \quad \mathbf{C}(k)\mathbf{x}(k) \quad \mathbf{v}(k) \tag{2}$$

where $\mathbf{z}$ is the m-dimensional measurement vector and $\mathbf{v}$ is the m-dimensional unknown measurement noise vector. The time-variable measurement matrix $\mathbf{C}(k)$ is $m \quad n(m \quad n)$-dimensional and has maximal rank. Note that as discussed above for $\mathbf{w}$, a probabilistic description of the noise $\mathbf{v}$ is not necessary. Given a sequence of measurements, $\mathbf{z}(0), \mathbf{z}(1), \ldots, \mathbf{z}(k)$, the unconstrained weighted linear least squares state estimate is generated by minimizing a weighted sum of the squares of the noises, that is, the performance index is

$$J_{k+1} = \frac{1}{2}\,{}^{k}_{j=0}\left(\mathbf{z}(j) \quad \mathbf{C}(j)\mathbf{x}(j)\right)^T \mathbf{R}^{\;1}(j)\left(\mathbf{z}(j) \quad \mathbf{C}(j)\mathbf{x}(j)\right) + \frac{1}{2}\,{}^{k}_{j=0}\left(\mathbf{x}(j+1) \quad A(j)\mathbf{x}(j)\right)^T \mathbf{Q}^{\;1}(j) \left(\mathbf{x}(j+1) \quad A(j)\mathbf{x}(j)\right) \tag{3}$$

where $\mathbf{R}(j)$ and $\mathbf{Q}(j)$ are positive definite weighting matrices. The state vectors $\mathbf{x}(0), \mathbf{x}(1), \ldots, \mathbf{x}(k), \mathbf{x}(k+1)$ are chosen to minimize J_{k+1}. The weighting matrices $\mathbf{R}$ and $\mathbf{Q}$ are design parameters. If the covariance matrices of $\mathbf{v}$ and $\mathbf{w}$ are known, they would be an appropriate choice for $\mathbf{R}$ and $\mathbf{Q}$, respectively. Otherwise, they must be chosen from some other criteria, with default values simply being identity matrices. The values of the state vectors that minimize J_{k+1} are denoted $\hat{\mathbf{x}}_u(0\,|\,k), \hat{\mathbf{x}}_u(1\,|\,k), \ldots, \hat{\mathbf{x}}_u(k\,|\,k), \hat{\mathbf{x}}_u(k+1\,|\,k)$, that is, these are the unconstrained estimates of the state vector at each time instant based on the measurements $\mathbf{z}(0), \mathbf{z}(1), \ldots, \mathbf{z}(k)$. In reference [5], it was shown that the unconstrained solution for the desired current estimate, $\hat{\mathbf{x}}_u(k\,|\,k)$, is generated sequentially from the following linear vector-matrix difference equation:

$$\hat{\mathbf{x}}_u(k\,|\,k) = \mathbf{A}(k \quad 1)\hat{\mathbf{x}}_u(k \quad 1\,|\,k \quad 1) + \mathbf{K}(k)\left(\mathbf{z}(k) \quad \mathbf{C}(k)\mathbf{A}(k \quad 1)\hat{\mathbf{x}}_u(k \quad 1\,|\,k \quad 1)\right) \tag{4}$$

with the initial condition $\hat{\mathbf{x}}_u(\; 1\,| \; 1) = 0$ and where

$$\mathbf{K}(k) = \mathbf{M_{11}}(k)\mathbf{C}^T(k)\mathbf{R}^{\;1}(k) \tag{5}$$

and

$$\mathbf{M_{11}}(k) = \left[\left(\mathbf{C}^T(k)\mathbf{R}^{\;1}(k)\mathbf{C}(k) + \mathbf{Q}^{\;1}(k \quad 1)\right) \quad \mathbf{W}(k)\right]^{\;1} \tag{6}$$

The matrix $\mathbf{W}(k)$ can be calculated from the matrix difference equation:

$$\mathbf{W}(k) = \mathbf{Q}^{\;1}(k \quad 1)\mathbf{A}(k \quad 1) \left[\left(\mathbf{C}^T(k \quad 1)\mathbf{R}^{\;1}(k \quad 1)\mathbf{C}(k \quad 1) + \mathbf{A}^T(k \quad 1)\mathbf{Q}^{\;1}(k \quad 1)\mathbf{A}(k \quad 1) + \mathbf{Q}^{\;1}(k \quad 2)\right) \quad \mathbf{W}(k \quad 1)\right]^{\;1} \mathbf{A}^T(k \quad 1)\mathbf{Q}^{\;1}(k \quad 1) \tag{7}$$

with the initial conditions $\mathbf{W}(0) = \mathbf{0}$ and $\mathbf{Q}^{\;1}(\; 1) = \mathbf{0}$.

Now assume a constraint is added that the state of the system is constrained at time k by an instantaneous algebraic inequality of the form:

$$f\left(\mathbf{x}(k), k\right) = \mathbf{d}^T\mathbf{x}(k) \qquad , \tag{8}$$

For brevity of discussion, we have assumed that the constraint is linear in the state, as shown. (The general constrained estimation problem in which the imposed constraints can be nonlinear is discussed in [5,6].) Following that discussion, if at time k, $\mathbf{d}^T\mathbf{x}(k) \qquad$, then the constrained estimate is $\hat{\mathbf{x}}(k\,|\,k) = \hat{\mathbf{x}}_u(k\,|\,k)$ and the unconstrained estimation process continues to the next time step. If $\mathbf{d}^T\mathbf{x}(k) >$, then the constrained estimate is determined by solving the following vector-matrix equation:

$$\left[\begin{array}{c:c} \mathbf{I} & \mathbf{M_{11}}(k)\mathbf{d} \\ \hdashline \mathbf{d}^T & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \hat{\mathbf{x}}(k \mid k) \\ \hdashline (\ (k) \end{array}\right] = \left[\begin{array}{c} \hat{\mathbf{x}}_u(k \mid k) \\ \hdashline \ ) \end{array}\right] \tag{9}$$

The final constrained solution is given by

$$\hat{\mathbf{x}}(k \mid k) = \hat{\mathbf{x}}_u(k \mid k) + \frac{\mathbf{M_{11}}(k)\mathbf{d}}{\mathbf{d}^T \mathbf{M_{11}}(k)\mathbf{d}}\left(\quad \mathbf{d}^T \hat{\mathbf{x}}_u(k \mid k)\right) \tag{10}$$

Now assume that the system is modeled by the nonlinear difference equation

$$\mathbf{x}(k+1) = \mathbf{a}(\mathbf{x}(k),k) + \mathbf{w}(k) \tag{11}$$

where $\mathbf{a}(\mathbf{x}(k),k)$ is a nonlinear, time-variable vector state transition function of dimension n, the measurement of the state is given by

$$\mathbf{z}(k) = \mathbf{c}(\mathbf{x}(k),k) + \mathbf{v}(k) \tag{12}$$

where $\mathbf{c}(\mathbf{x}(k),k)$ is a nonlinear, time-variable vector measurement of dimension m, the performance index is given by

$$\begin{aligned} J_{k+1} = & \\ & \frac{1}{2}\sum_{j=0}^{k} \left(\mathbf{z}(j) \quad \mathbf{c}(\mathbf{x}(j),j)\right)^T \mathbf{R}^{\ 1}(j)\left(\mathbf{z}(j) \quad \mathbf{c}(\mathbf{x}(j),j)\right) \\ & + \frac{1}{2}\sum_{j=0}^{k} \left(\mathbf{x}(j+1) \quad \mathbf{a}(\mathbf{x}(j),j)\right)^T \mathbf{Q}^{\ 1}(j) \\ & \qquad \left(\mathbf{x}(j+1) \quad \mathbf{a}(\mathbf{x}(j),j)\right) \end{aligned} \tag{13}$$

and the scalar nonlinear state constraint is given by

$$f\left(\mathbf{x}(k),k\right) \tag{14}$$

Using a quasi-linearization technique in which whenever a nonlinearity is expanded in a Taylor series, the linear term of the expansion is retained and evaluated about the last best estimates of the states and measurements, an extended version of the linear time-variable sequential estimator can be developed which generates an unconstrained estimate of the state for the given nonlinear system. This estimate is given by

$$\begin{aligned} \hat{\mathbf{x}}_u(k \mid k) = & \ \mathbf{a}(\hat{\mathbf{x}}_u(k \quad 1 \mid k \quad 1), k \quad 1) + \\ & \mathbf{K}(k)\left[\mathbf{z}(k) \quad \mathbf{c}(\mathbf{a}(\hat{\mathbf{x}}_u(k \quad 1 \mid k \quad 1), k \quad 1))\right] \end{aligned} \tag{15}$$

where the filter gain $\mathbf{K}(k)$ is calculated from Eqs. (5), (6), and (7) in which

$$\mathbf{A}(k) = \left.\frac{\mathbf{a}(\mathbf{x},k)}{\mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_u(k|k \quad 1)}$$

and

$$\mathbf{C}(k) = \left.\frac{\mathbf{c}(\mathbf{x},k)}{\mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_u(k|k \quad 1)}$$

The constrained nonlinear estimate is then determined as follows.

If $f\left(\hat{\mathbf{x}}_u(k \mid k)\right)$, then $\hat{\mathbf{x}}(k \mid k) = \hat{\mathbf{x}}_u(k \mid k)$.

If not, $\hat{\mathbf{x}}_u(k \mid k) = \mathbf{K}(k) \quad \mathbf{z}(k)$ is calculated from the second term on the right side of Eq. (14) and the constrained estimate, $\hat{\mathbf{x}}(k \mid k)$, of $\mathbf{x}(k)$ is calculated by applying the weighted linear least squares time-variable results from Eqs. (4)-(7) and (10) to the constrained estimation problem for the linearized system equations

$$\mathbf{x}(k+1) = \mathbf{A}(k) \quad \mathbf{x}(k) + \mathbf{w}(k), \quad k \quad 0$$

$$\mathbf{z}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{v}(k)$$

This results in the constrained estimate

$$\begin{aligned} \hat{\mathbf{x}}(k \mid k) = & \quad \hat{\mathbf{x}}_u(k \mid k) \\ & + \frac{\mathbf{M_{11}}(k)\mathbf{F}(k)}{\mathbf{F}^T(k)\mathbf{M_{11}}(k)\mathbf{F}(k)}\left(\ ' \quad \mathbf{F}^T(k) \quad \hat{\mathbf{x}}_u(k \mid k)\right) \end{aligned} \tag{16}$$

where

$$' = \quad f(\hat{\mathbf{x}}_u(k \mid k \quad 1), k)$$

and

$$\mathbf{F}^T(k) = \frac{f(\hat{\mathbf{x}}_u(k \mid k \quad 1))}{\hat{\mathbf{x}}_u(k \mid k \quad 1)}$$

and where $\hat{\mathbf{x}}_u(k \mid k \quad 1) = \mathbf{a}(\hat{\mathbf{x}}_u(k \quad 1 \mid k \quad 1), k \quad 1)$ is obtained as the first term on the right side of Eq. (15). Finally then, the nonlinear estimate of the system state is given by

$$\hat{\mathbf{x}}(k \mid k) = \hat{\mathbf{x}}_u(k \mid k \quad 1) + \quad \hat{\mathbf{x}}(k \mid k).$$

3 Incorporation Of The Constraint Estimator With The Extended Kalman Filter

The estimation technique used as the heart of most Level 1 kinematic sensor fusion techniques is that of the extended Kalman filter [7]. For the sequential constrained estimator described above to be implemented in a tracking system, either the EKF needs to be replaced by the new estimation routine or it can be added as a separate estimator used in a VS-IMM. The former implementation would require acceptance of equivalence or superiority in performance of the sequential constrained estimator when compared to the EKF. The latter technique would require more research.

Another approach to implementing the constrained estimator is to note that in the computation of the (linear and nonlinear) constrained estimate, Eq. (10) and Eq.(17), the estimator only requires the current unconstrained estimate. Also, the unconstrained estimate, in each case, is not dependent on the constrained estimate as we see in Eq.(4) and Eq.(15). This decoupling of the two parts of the estimator allow the unconstrained estimator to be replaced by the EKF.

We can construct the hybrid estimation technique as follows. The unconstrained estimate is generated from a standard set of the EKF equations [8]. Starting at time k with the unconstrained estimate $\mathbf{x}_u(k \mid k)$, the first step is the prediction step.

$$\hat{\mathbf{x}}_u(k \mid k \quad 1) = \mathbf{a}(\hat{\mathbf{x}}_u(k \quad 1 \mid k \quad 1), k \quad 1)$$

$$\begin{aligned} \mathbf{P}(k \mid k \quad 1) = & \\ & \overline{\mathbf{A}}(k \quad 1)\mathbf{P}(k \quad 1 \mid k \quad 1)\overline{\mathbf{A}}^T(k \quad 1) + \mathbf{Q}(k) \end{aligned}$$

where

$$\overline{\mathbf{A}}(k \quad 1) = \left.\frac{\mathbf{a}(\mathbf{x},k \quad 1)}{\mathbf{x}}\right|^T_{\mathbf{x}=\hat{\mathbf{x}}_u(k \quad 1|k \quad 1)}$$

Note that this matrix is different from $\mathbf{A}(k \quad 1)$ defined earlier.

The next step is to update the estimate using the the new data at time k as follows.

$$\mathbf{K}(k) = \mathbf{P}(k \mid k \quad 1)\mathbf{C}(k)^T$$
$$\left(\mathbf{C}(k)\mathbf{P}(k \mid k \quad 1)\mathbf{C}(k)^T + \mathbf{R}(k)\right)^{\ 1}$$

where $\mathbf{C}(k)$ is not different from $\mathbf{C}(k)$ defined above. Finally,

$$\hat{\mathbf{x}}_u(k \mid k) =$$
$$\hat{\mathbf{x}}_u(k \mid k \quad 1) + \mathbf{K}(k)\left(\mathbf{z}(k) \quad \mathbf{c}(\hat{\mathbf{x}}_u(k \mid k \quad 1), k)\right)$$

$$\mathbf{P}(k \mid k) = \left(\mathbf{I} \quad \mathbf{K}(k)\mathbf{C}(k)\right)\mathbf{P}(k \mid k \quad 1)$$

Finally, using the constraint boundary equation, the constrained update is formed from the following equation.

$$\frac{\hat{\mathbf{x}}_c(k \mid k)}{(k)} = \begin{array}{c|c} \mathbf{I} & (\ \mathbf{M_{11}}(k\ (\ 1)\mathbf{F}(k)\ ^{(1} \\ \hline \mathbf{F}(k) & \mathbf{0} \end{array} \qquad (18)$$
$$)\ \frac{\hat{\mathbf{x}}_u(k \mid k)}{(\ f_k\left(\hat{\mathbf{x}}_u(k \mid k)\right) + \mathbf{F}(k)\hat{\mathbf{x}}_u(k \mid k)}$$

where

$$\mathbf{F}(k) = \frac{f_k}{\mathbf{x}}\bigg|_{\mathbf{x} = \hat{\mathbf{x}}_u(k|k)}$$

The constraint equation presented in Eq. 18 is similar to that of Eq. 9. In Eq. 9, the constraint is assumed to be linear. In Eq. 18, we have linearized the constraint. As described, the constraint is linearized about the unconstrained update. We have also tested the routine with the constraint linearized about a predicted unconstrained estimate and a predicted constrained estimate. As with the EKF, the point of linearization is an important factor in the calculation. The further that the linearization point is away from the constraint, the worse the linearization is. All of these techniques have been shown to provide adequate linearization in our test cases. The predicted unconstrained estimate as the linearization point is similar to the technique that was derived in the development of the constrained sequential estimator [6].

4 Comparison Example 1

Our first example compares the standard implementation of the sequential constraint estimator to the implementation which employs the EKF for the unconstrained estimator. The measurements are linearly related to the states. The constraint is nonlinear. This requires a linearization of the constraint in the constrained estimate equations as in Eq. 16 and Eq. 18 for the standard implementation and the EKF implementation, respectively.

The constraint function was defined as

$$x^2 + 4y \quad 3.$$

The dynamic system used in the Kalman filter was defined as

$$\mathbf{x}(k+1) = \begin{matrix} 1 & dt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 1 \end{matrix}\ \mathbf{x}(k)$$

with the position components of the state being measured directly

$$\mathbf{z}(k) = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}\ \mathbf{x}(k)$$

The covariance of the process noise was defined as

$$\mathbf{Q}_k = \begin{matrix} dt^2 & dt & 0 & 0 \\ dt & 1 & 0 & 0 \\ 0 & 0 & dt^2 & dt \\ 0 & 0 & dt & 1 \end{matrix}\ ^2$$

where dt is defined as 1.0 seconds and = 0.5. The measurement noise was defined as I_{2x2}. Our measurements, which are given as x, y positions, were defined as $[0,0]$, $[1,2]$, $[2,0]$, $[3,\ 2]$, $[4,\ 4]$, and $[5,\ 7]$.

Figure 2 shows the measurements and the constraint. We see that the second and third measurement are clearly outside the constraint region. In Figure 3, we see the output of the constrained estimator when an EKF is used as the unconstrained estimator. The +'s incicate the constrained estimates while the o's denote the unconstrained estimates. Visually, the constrained estimates are corrected. Numerically, the estimates are significantly improved over the unconstrained estimates. We compare these results to Figure 4 which is the result of the standard sequential estimator with the constraint. Again, the results show significant improvement of the constrained estimator over the unconstrained estimates. Figure 3 clearly indicates in this one test case, the use of the EKF as the unconstrained estimator has a better performance in that it is closer to the constraint.

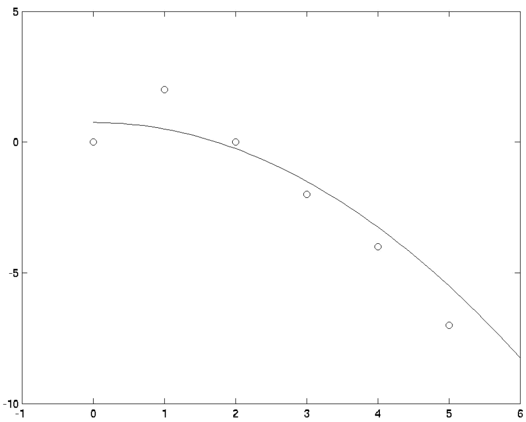

Figure 2: The constraint and the measurements for our comparison case are shown.

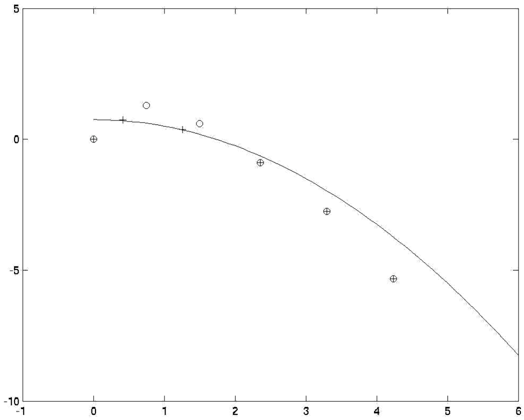

Figure 3: Using the Kalman filter as the unconstrained estimator provides a working constraint estimation routine.

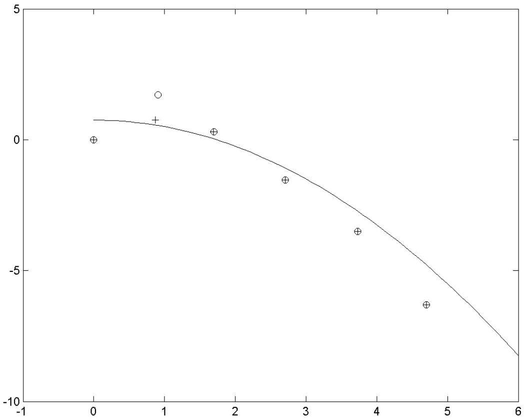

Figure 4: The constrained sequential estimator routine also improves the reported state estimate.

In Table 1, we present the coordinates of the position for the constrained and unconstrained estimates of both techniques. Clearly, we see that both unconstrained estimates are quality estimates with the baseline estimator following closer the measurements. The constrained estimates are both closer to the constraint with the EKF technique closer to the constraint than the standard sequential estimator. This is a result of the linearization steps in each approach. The standard approach linearizes the constraint about the predicted estimate. The EKF approach linearizes the constraint about the updated estimate.

5 Comparison Example 2

Our second comparative example is a more complex example. The track state is represented by a six state system (position, velocity, and acceleration) of a two dimensional tracking problem. The truth track is described as a set of locations that obeys the constraint. The measurements are defined as a three element vector, range and the cosine and sine of the azimuth angle. The measurements then have Gaussian noise added to them.

A number of cases were run with different noise seeds. In some cases, all of the measurements were within the constraint boundary which in this case is linear. For this effort, we chose an example that clearly demonstrated that the constraint was violated. The constraint for this case was defined as

$$2.2x + y \quad 0.$$

Figure 5 shows a portion of the target trajectory (dashed line) when it gets near the constraint (solid line). The noisy measurements, indicated by o's clearly show that they are reported outside the constraint. Figure 6 depicts the EKF version of the constraint estimation approach. The +'s indicate the constrained estimates while the o's indicate the unconstrained estimates. The solid line depicts the boundary of the constraint. We see that the constraint is violated by the unconstrained estimate. The constrained estimator returns the estimate to the feasible space. Figure 7 presents the same results for the baseline sequential constraint estimator. The dashed line indicates the true trajectory from which the noisy measurements were generated. We see both techniques return the reported estimate to the feasible space based on the concept of minimizing their respective cost functions.

Table 1: Constrained and unconstrained estimates for both techniques

Type of Estimate	*k*	*Baseline Technique*	*EKF Technique*
	1	0, 0	0, 0
	2	0.906, 1.712	0.745, 1.287
	3	1.698, 0.309	1.501, 0.605
Unconstrained	4	2.709, -1.537	2.356, -0.908
	5	3.728, -3.497	3.299, -2.765
	6	4.704, -6.308	4.236, -5.324
	1	0, 0	0, 0
	2	0.869, 0.750	0.414, 0.774
Constrained	3	1.694, 0.298	1.263, 0.370
	4	2.709, -1.537	2.356, -0.908
	5	3.728, -3.497	3.299, -2.765
	6	4.704, -6.308	4.236, -5.324

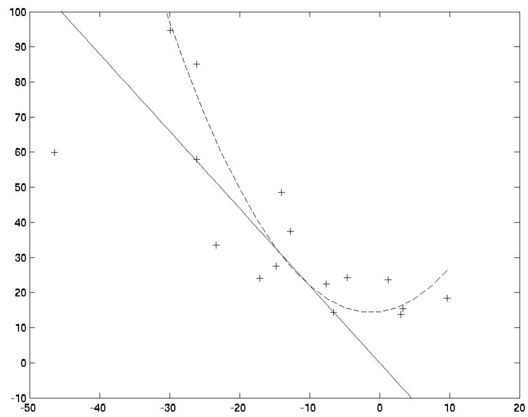

Figure 4: The target's trajectory places it near the constraint. The noise on the measurements causes the estimate to violate the constraint.

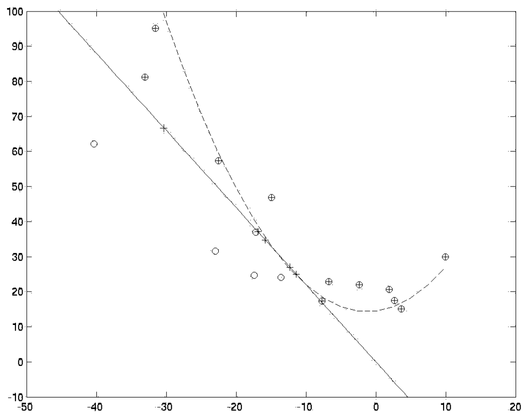

Figure 5: The EKF approach to constraint estimation works well.

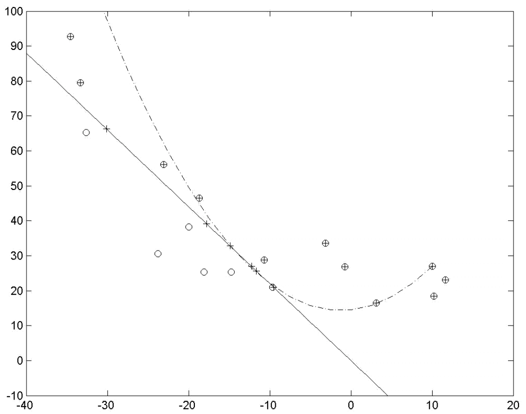

Figure 6: The baseline system provides a quality constraint estimate of the target's trajectory.

6 Conclusions

In this paper, we have demonstrated that the previously developed sequential constrained estimator can be modified to work with the extended Kalman filter in estimation. Unlike some ad hoc constrained estimates, the technique provided in this paper is designed to minimize a mean squared error cost function. Thus, placement of the estimate in the feasible space is a mathematical approach with a solid foundation. The technique can be used in conjunction with current filters as an added step prior to reporting the track to the operator.

We have also discover through the work of this effort some more areas of research. First, the mathematics that developed the constrained estimator required the use of the predicted unconstrained estimate to be used for linearizing the constraint. We have begun to look at the mathematics to see if the updated estimate as the linearization point will still allow us to optimize the cost function. This could allow for improved performance for nonlinear constraints. We also need to perform comparisons of the unconstrained estimator to that of the Kalman filter in performance. The basic technique should be benchmarked against the standard estimation technique.

Finally, we plan to expand our implementation and testing to a wider variety of problems. We plan to look at more tracking problems including using the technique in conjunction with an IMM. We also plan to investigate the performance with multiple constraints that could occur in group tracking. Also, other applications outside of tracking including estimation with quaternions in which a normalization step is generally added to guarantee that the estimates of the quaternions satisfy their magnitude constraints.

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